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A LIFTING-SURFACE THEORY FOR VISCOUS FLOW

Fujio Yamamoto*

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In the present paper, a new lifting-surface theory for viscous flow is described. The theory is based on the Imai's elementary solutions of the three-dimensional incompressible Oseen flow. The lifting force on a flat plate in the viscous flow can be calculated by means of the theory which consists of the following three steps: (1) The first step is to find approximately the vorticity distribution by means of the lifting-surface theory of D.E.Davies on the assumption of invicid flow, (2) the second is to calculate the induced velocity due to the viscous component of the kernel for the vorticity distribution found in the first step, (3) the third is to calculate the vorticity distribution and the lifting force applied on a flat plate by means of the D.E.Davies's method, after modifying the induced velocity of the boundary conditions by substitution of that estimated in the second step. The effects of viscosity of fluid and of aspect ratio of a rectangular flat plate on the lifting force are discussed by using some computational results.

1. Introduction

Although there have been developed many lifting-surface theories on the assumption of invicid flow and they are available to discuss the lifting force on the wing in consideration of unsteadiness and compressibility of the flow, there have been very few lifting-surface theories which are available to discuss the effects of viscosity of fluid on the lifting force applied on the finite span wing.

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The papers by S.N.Brown et al.[1] and by S.Murata et al.[2] discuss the effects of viscosity on the unsteady force applied on the vibrating flat plate. The former develops the theory based on the boundary layer approximation and the latter based on the Oseen approximation. If the former method is extended and applied to a three-dimensional wing, the expressions may become too complicated. On the other hand, the Oseen approximation possesses the merit that the expression can desribe the whole field of flow with only one linear equation and it may be said that the Oseen approximation is useful to discuss the effects of viscosity on the lifting force. Recently, S.Murata et al.[3] have published the elementary solutions of the three-dimensional Oseen flow. It may be possible to establish a lifting-surface theory for unsteady and viscous flow by making use of their solutions, and it will be discussed in the second paper.

In the present paper, a lifting-surface theory based on the Imai's elementary solution of the three-dimensional steady incompressible Oseen flow is proposed. The effects of viscosity and of the aspect ratio of the rectangular flat plate on the lifting force are discussed by the aid of some examples of numerical calculation.

2. Lifting-Surface Theory

2.1 Integral Equation

Consider the steady incompressible flow of fluid with perturbation velocity $\boldsymbol{v}$, perturbation pressure p, density ρ, kinematic viscosity ν, when a small concentrated force is applied at the original point in an otherwise uniform flow with velocity U in the x direction. Applying the Oseen approximation to the flow, the fundamental equations are as follows:

$$\mathrm{div}\,\boldsymbol{v} = 0 \tag{1}$$

$$U\frac{\partial \boldsymbol{v}}{\partial x} = -\frac{1}{\rho}\,\mathrm{grad}\,p + \nu\nabla^2\boldsymbol{v} \tag{2}$$

K.Imai[4] published the elementary solution of the above equations. The solution expresses the induced velocity due to an Oseenlet with strength $4\pi\rho U\boldsymbol{A}$ which is put on the original point, where $\boldsymbol{A}$ denotes one of unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ in x, y, z directions, respectively. The solution is expressed as follows:

$$\boldsymbol{v} = \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{A}\,(\chi_f - \phi_f) \tag{3}$$

where

$$\chi_f = E\left[k\,(r_1 - x)\right] \tag{4}$$

$$E(\xi) = \int_{\infty}^{\xi}\frac{e^{-\zeta}}{\zeta}\,d\zeta \tag{5}$$

$$\phi_f = \log(r_1 - x) \tag{6}$$

$$r_1 = x^2 + y^2 + z^2 \tag{7}$$

$$k = \frac{U}{2\nu} \tag{8}$$

Let the induced velocity due to the Oseenlet with the axis in the z direction be $v^{z)}$, it becomes as follows after putting $A = K$:

$$v^{z)} = -\frac{z}{r_1^2}\left[\frac{1}{r_1}\left\{1 - e^{-k(r_1-x)}\right\} - k e^{-k(r_1-x)}\right] i - \frac{1}{r_1}\left[\frac{1}{r_1 - x}\left\{1 - e^{-k(r_1-x)}\right\} - 2k e^{-k(r_1-x)}\right] K + \frac{r_1 - x}{r_1^2}\,\frac{z}{y^2+z^2}\left[\left(\frac{1}{r_1} + \frac{1}{r_1 - x}\right)\left\{1 - e^{-k(r_1-x)}\right\} - k e^{-k(r_1-x)}\right] \frac{y j - zK}{\sqrt{y^2+z^2}} \tag{9}$$

Next, the lifting force on a rectangular flat plate with infinitely small thickness which is immersed in the uniform flow will be discussed. The co-ordinate axes are taken as shown in Fig.1. Let the chord length be c and span s. In case of distributions of Oseenlets with the dimensionless strength $\gamma(x_0, y_0)$ on the wing surface, the induced velocity $V = ui + vj + wK$ at a point $(x, y, 0)$ on the wing surface is as follows after putting $z = 0$ in Eq.(9):

$$u = v = 0 \tag{10}$$

$$\alpha(x, y) = -\frac{w}{U} = \int_{-\frac{s}{2}}^{+\frac{s}{2}} \gamma(x_0, y_0)\, K(x - x_0,\ y - y_0)\, dx_0\, dy_0 \tag{11}$$

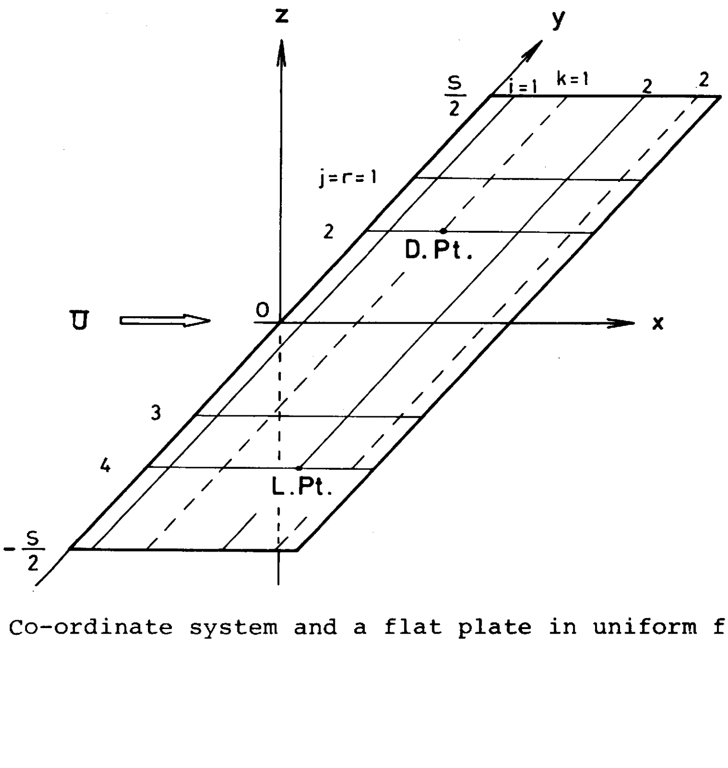

Fig.1 Co-ordinate system and a flat plate in uniform flow

Here α denotes the induced incident angle and it decides the boundary conditions. From Eq.(9), the kernel K in the above equation becomes as follows:

$$K(x-x_0,\ y-y_0) = -\frac{1}{R}\left[\frac{1}{R-X}\left\{1-e^{-k(R-X)}\right\}-2ke^{-k(R-X)}\right] \tag{12}$$

where

$$R^2 = X^2+Y^2\ ,\quad X = x-x_0\ ,\quad Y = y-y_0 \tag{13}$$

There is a relationship between vorticity distribution $\gamma(x_0, y_0)$ and pressure difference $\Delta p(x_0, y_0)$ at a point $(x_0, y_0, 0)$ on the wing surface as follows:

$$\gamma(x_0, y_0) = \frac{\Delta p(x_0, y_0)}{4\pi\rho U^2} \tag{14}$$

The relationship among lifting force L , Δp and γ is given as follows:

$$\begin{aligned} L &= \int_{-\frac{s}{2}}^{+\frac{s}{2}}\int_0^c \Delta p(x_0, y_0)\, dx_0\, dy_0 \\ &= 4\pi\rho U^2\int_{-\frac{s}{2}}^{+\frac{s}{2}}\int_0^c \gamma(x_0, y_0)\, dx_0\, dy_0 \end{aligned} \tag{15}$$

Therefore, after solving the integral equation (11) about γ, L can be calculated from Eq.(15).

2.2 Three Steps for Calculating the Lifting Force

In order to solve the integral equation (11) and to calculate the lifting force, a new lifting-surface theory should be devised in consideration of viscosity. The kernel (12) can be divided into the invicid component K_p and the viscous component K_v as follows:

$$K(X,Y) = K_p(X,Y) + K_v(X,Y) \tag{16}$$

$$K_p(X,Y) = -\frac{1}{R(R-X)} = \frac{K_p^*(X,Y)}{Y^2} \tag{17}$$

where

$$K_p^*(X,Y) = -\left(1+\frac{X}{R}\right) \tag{18}$$

$$K_v(X,Y) = \frac{1}{R(R-X)}e^{-k(R-X)} + \frac{2k}{R}e^{-k(R-X)} \tag{19}$$

H.Multhopp[(5)] and D.E.Davies[(6)] published the lifting-surface theories. They applied the methods of interpolation function to K_p after changing Eq. (11) into the first order principal value integral and evaluating the logarithmic singularities of the integral of K_p . However, if their methods are applied directly to the case of viscous fluid, the logarithmic singularities and K_v are extremely effected by Reynolds number $R_e = kc$. Therefore, no interpolation functions are valid for the wide range of Re. Now, a lifting-surface theory for viscous flow, which is subjective to the range of $Re \geq 10$, will be proposed. The range of $Re \geq 10$ seems to have comparatively small effects of viscosity. The theory includes the next three steps for calculation of the lifting force;

(i) First, assuming the invicid flow, $\gamma = \gamma_p$ for the invicid component of the kernel is calculated by means of D. E. Davies's method after putting $K = K_p = Y^2 K_p^*$ and $\alpha = 1$ in Eq.(11).

(ii) Putting $\gamma = \gamma_p$, $K = K_v$ in Eq.(11), the induced incident angle α_v due to the viscous component of the kernel is calculated.

(iii) Putting $\alpha_p = \alpha - \alpha_v$, where α_p denotes the induced incident angle due to the invicid component of the kernel, γ is calculated by means of the lifting-surface theory of D.E.Davies, and then the lifting force L is calculated from Eq.(15).

2.2.1 Calculation of γ_p by Lifting-Surface Theory for Invicid Flow

The lifting-surface theory for invicid flow is applied to the steps of (i) and (iii) mentioned in the preceding section. In the present section, how to apply the theory of D. E. Davies[6] for invicid unsteady supersonic flow to the incompressible steady flow is described.

The dimensionless co-ordinates are introduced for a flat plate with chord length c and span s as follows:

$$\left.\begin{aligned} \xi &= \frac{x}{c}, \quad \eta = \frac{y}{s/2} \\ \xi_0 &= \frac{x_0}{c}, \quad \eta_0 = \frac{y_0}{s/2} \end{aligned}\right\} \tag{20}$$

Moreover, putting

$$\bar{\gamma}_p(\xi_0, \eta_0) = \gamma_p(x_0, y_0) \tag{21}$$

$$\bar{\alpha}_p(\xi, \eta) = \alpha_p(x, y) \tag{22}$$

and employing Eq.(17), Eq.(11) is rewrittten for the case of the invicid flow as follows:

$$\bar{\alpha}_p(\xi, \eta) = \frac{2c}{s}\int_{-1}^{+1} \frac{1}{(\eta-\eta_0)^2}\, \gamma_p(\xi_0, \eta_0)\, K_p^*(X,Y)\, d\xi_0\, d\eta_0 \tag{23}$$

The downwash points (ξ_k, η_r) and loading points (ξ_i, η_j) are selected as follows:

$$\left.\begin{aligned} \xi_k &= \frac{1-\cos\theta_k}{2}, & \theta_k &= \frac{2k}{2n+1}\pi & (k &= 1, 2, ---, n) \\ \eta_r &= \cos\phi_r, & \phi_r &= \frac{r}{m+1}\pi & (r &= 1, 2, ---, m) \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} \xi_i &= \frac{1-\cos\theta_i}{2}, & \theta_i &= \frac{2i-1}{2n+1}\pi & (i &= 1, 2, ---, n) \\ \eta_j &= \cos\phi_j, & \phi_j &= \frac{j}{m+1}\pi & (j &= 1, 2, --- m) \end{aligned}\right\} \tag{25}$$

where n and m denote the number of the chordwise and spanwise directions, respectively. Let's express the continuous loading distributions of $\bar{\gamma}_p(\xi_0,\eta_0)$ by using the discontinuous loading distributions $\bar{\gamma}_p(\xi_i,\eta_j)$ and interpolation functions as follows:

$$\bar{\gamma}_p(\xi_0,\eta_0) = \sum_{i=1}^{n}\sum_{j=1}^{m}\bar{\gamma}_p(\xi_i,\eta_j)\, h_i^{(n)}(\xi_0)\, g_j^{(m)}(\eta_0) \tag{26}$$

In the above equation the interpolation function of $h_i^{(n)}(\xi_0)$ has the singularities of minus half power at the leading edge of flat plate and $g_j^{(m)}(\eta_0)$ has no singularities at the wing tips. These functions were employed by D.E.Davies[6], and are of the forms

$$\left.\begin{aligned} h_i^{(n)}(\xi_0) &= \frac{2(-1)^{i+1}}{2n+1}\sin\theta_i \sin\frac{\theta_i}{2}\,\frac{\cos(n+\frac{1}{2})\theta_0}{\sin\frac{\theta_0}{2}}\,\frac{1}{\cos\theta_0-\cos\theta_i} \\ g_j^{(m)}(\eta_0) &= \frac{(-1)^{j+1}}{m+1}\sin\phi_j\,\frac{\sin(m+1)\phi_0}{\cos\phi_0-\cos\phi_j} \end{aligned}\right\} \tag{27}$$

where

$$\xi_0 = \frac{1-\cos\theta_0}{2}, \qquad \eta_0 = \cos\phi_0 \tag{28}$$

Substituting Eqs.(24) to (28) into Eq.(23), modifying the integral equation with the singularities of $1/r^2$ into the first order principal value integral by partial integration and evaluating the logarithmic singularities, the incidence distributions are expressed as follows:

$$\begin{aligned} \bar{\alpha}_p(\xi_k,\eta_r) = {} & \sum_{i=1}^{n}\frac{\bar{\gamma}_p(\xi_i,\eta_r)}{\sqrt{1-\eta_r^2}}\,F_{0,i}(\xi_k)\left[A(\eta_r)-B(\eta_r)\right] \\ & + \sum_{i=1}^{n}\sum_{j=1}^{m}\bar{\gamma}_p(\xi_i,\eta_j)\, I_i^{(n)}(\eta_r,\eta_j,\xi_k)\,P_{j,r}^{(m)} \end{aligned} \tag{29}$$

where

$$A(\eta_r) = \int_{+1}^{-1}\log|\eta_r-\eta_0|\sqrt{1-\eta_0^2}\,d\eta_0 = \frac{\pi}{4}(2\eta_r^2-1)-\frac{\pi}{2}\log 2 \tag{30}$$

$$B(\eta_r) = \sum_{j=1}^{m}(\eta_r-\eta_j)^2\log|\eta_r-\eta_j|\sqrt{1-\eta_j^2}\,P_{j,r}^{(m)} \tag{31}$$

$$P_{j,r}^{(m)} = \int_{-1}^{+1}\frac{g_j^{(m)}(\eta_0)}{(\eta_r-\eta_0)^2}\,d\eta_0 = \begin{cases} -\dfrac{\pi}{2}\,\dfrac{m+1}{\sqrt{1-\eta_j^2}} & (r=j) \\ \dfrac{2\pi}{m+1}\,\dfrac{\sqrt{1-\eta_j^2}}{(\eta_r-\eta_j)^2} & (r+j=odd) \\ 0 & (r+j=even \text{ and } r\neq j) \end{cases} \tag{32}$$

$$F_{0,i}(\xi_k) = \frac{S^2}{4c^2}\,h_i^{(n)\prime}(\xi_k) \tag{33}$$

$$\begin{aligned} h_i^{(n)\prime}(\xi_k) &= \left[\frac{d}{d\xi_0}\left\{h_i^{(n)}(\xi_0)\right\}\right]_{\xi_0=\xi_k} \\ &= h_i^{(n)}(\xi_k)\left[-(n+\tfrac{1}{2})\tan(n+\tfrac{1}{2})\theta_k-\tfrac{1}{2}\cot\frac{\theta_k}{2}+\frac{\sin\theta_k}{\cos\theta_k-\cos\theta_i}\right]\frac{2}{\sin\theta_k} \end{aligned} \tag{34}$$

$$I_i^{(n)}(\eta_r, \eta_j, \xi_k) = \frac{2c}{s}\int_0^1 h_i^{(n)}(\xi_0) K_p^*(x_k - x_0,\ y_r - y_j)\, d\xi_0 \tag{35}$$

The influence function $I_i^{(n)}$ has no singularities and it can be calculated easily by means of a numerical method. When $\eta_r = \eta_j$,

$$K_p^*(X, 0) = -\left(1 + \frac{X}{R}\right) = \begin{cases} -2 & (X > 0) \\ 0 & (X < 0) \end{cases} \tag{36}$$

and then

$$I_i^{(n)}(\eta_r, \eta_j, \xi_k) = -\frac{4c}{s} h_i^{(1,n)}(\xi_k) \tag{37}$$

are obtained. Here,

$$h_i^{(1,n)}(\xi_k) = \int_0^{\xi_k - 0} h_i^{(n)}(\xi_0)\, d\xi_0 \tag{38}$$

Consequently, when $\bar{\alpha}_p(\xi_k, \eta_r)$ are given as the boundary conditions, Eq.(29) can be taken for the simultaneous linear equations about nm unknowns of $\bar{\gamma}_p(\xi_i, \eta_j)$, and they can be solved easily.

2.2.2 Calculation of α_v Due to Viscous Component

The present section deals with the step (ii) mentioned in section 2.2. Let the induced velocity due to the viscous component of the kernel be w_v and the induced angle α_v. If the loading distribution of γ can be replaced approximately by the distribution of γ_p, which has been calculated in the first step (i), we obtain

$$\alpha_v(x, y) = \frac{w_v}{U} = \int_{-\frac{s}{2}}^{+\frac{s}{2}} \gamma_p(x_0, y_0) K_v(x - x_0, y - y_0)\, dx_0 dy_0 \tag{39}$$

The viscous component of the kernel K_v is divided into two as follows:

$$K_v(X, Y) = K_{v1}(X, Y) + K_{v2}(X, Y) \tag{40}$$

$$K_{v1}(X, Y) = \frac{1}{R(R - X)} e^{-k(R-X)} = \frac{1}{Y^2} K_{v1}^*(X, Y) \tag{41}$$

where

$$K_{v1}^*(X, Y) = \left(1 + \frac{X}{R}\right) e^{-k(R-X)} \tag{42}$$

$$K_{v2}(X, Y) = \frac{2k}{R} e^{-k(R-X)} \tag{43}$$

Substituting Eqs.(40) to (43) into Eq.(39) and modifying the integral equation involving the singularities of $1/Y^2$ in the viscous component K_{v1} into the first order principal value integral equation by partial integration, we obtain

$$\alpha_v(X, Y) = \int_0^c \left[\left[\frac{1}{Y}\gamma_p(x_0, y_0) K_{v1}^*(X, Y)\right]_{y_0 = -\frac{s}{2}}^{y_0 = +\frac{s}{2}} - \int_{-\frac{s}{2}}^{+\frac{s}{2}} \left[\frac{1}{Y}\frac{\partial}{\partial y_0}\left\{\gamma_p(x_0, y_0) K_{v1}^*(X, Y)\right\} + \right.\right.$$

$$+\gamma_p(x_0, y_0)\, Kv_2(X,Y)\Big]dy_0\Big]dx_0 \tag{44}$$

The loading distribution of $\gamma_p(x_0, y_0)$ has nearly a trapezoidal shape in the Y direction. That is, it takes a flat shape in the middle of the span and approaches to zero with steep slopes at the wing tips. Thus, we can put

$$\gamma_p(x_0, \pm\frac{s}{2}) = 0 \tag{45}$$

$$\frac{\partial}{\partial y_0}\{\gamma_p(x_0, y_0)\} \doteqdot 0 \qquad (y_0 \neq \pm\frac{s}{2}) \tag{46}$$

Consequently, Eq.(44) becomes

$$\alpha_v(x, y) = \int_0^c \int_{-\frac{s}{2}}^{+\frac{s}{2}} \gamma_p(x_0, y_0)\, Kv_{12}(X,Y)\, dy_0\, dx_0 \tag{47}$$

where

$$Kv_{12}(X,Y) = \left\{k(R-X) - \frac{X}{R}\right\}\frac{1}{R^2}e^{-k(R-X)} \tag{48}$$

Assuming that the discontinuous distributions of $\bar{\gamma}_p$ may be replaced by the linear distributions in the spanwise direction and by the interpolation function $h_i^{(n)}(\xi_0)$ in the chordwise direction, $\bar{\gamma}_p$ is expressed as follows:

$$\bar{\gamma}_p(\xi_0, \eta_0) = \sum_{i=1}^{n}\sum_{j_1=0}^{m} \bar{\lambda}_{i,j_1}(\eta_0)\, h_i^{(n)}(\xi_0) \tag{49}$$

where

$$\bar{\lambda}_{i,j_1}(\eta_0) = \frac{\bar{\gamma}_p(\xi_i, \eta_{j_1+1}) - \bar{\gamma}_p(\xi_i, \eta_{j_1})}{\eta_{j_1+1} - \eta_{j_1}}(\eta_0 - \eta_{j_1}) + \bar{\gamma}_p(\xi_i, \eta_{j_1}) \tag{50}$$

$$\left.\begin{array}{lll} \eta_{j_1} = +1, & \bar{\gamma}_p(\xi_i, \eta_{j_1}) = 0 & (j_1 = 0) \\ \eta_{j_1} = \eta_j, & \bar{\gamma}_p(\xi_i, \eta_{j_1}) = \bar{\gamma}_p(\xi_i, \eta_j) & (j_1 = j = 1, 2, \cdots, m) \\ \eta_{j_1} = -1, & \bar{\gamma}_p(\xi_i, \eta_{j_1}) = 0 & (j_1 = m+1) \end{array}\right\} \tag{51}$$

Substituting Eq.(49) into Eq.(47) and introducing the dimensionless co-ordinates, the downwash points and loading points which were expressed by Eqs.(20), (24) and (25), respectively, we obtain

$$\alpha_v(\xi_k, \eta_r) = \frac{cs}{2}\sum_{i=1}^{n}\left(\int_0^{\xi_k-\delta} + \int_{\xi_k-\delta}^{\xi_k+\delta} + \int_{\xi_k+\delta}^{+1}\right)\Big[h_i^{(n)}(\xi_0)\sum_{j_1=0}^{m}\left(\int_{-1}^{\eta_r-\varepsilon} + \int_{\eta_r-\varepsilon}^{\eta_r+\varepsilon} + \int_{\eta_r+\varepsilon}^{+1}\right)$$
$$\bar{\lambda}_{i,j_1}(\eta_0)\, Kv_{12}(X,Y)\, d\eta_0\Big]\, d\xi_0 \tag{52}$$

Here δ and ε denote the dimensionless small lengths in the chordwise and spanwise directions, respectively. Integrating the right hand of Eq.(52), nm induced incident angles $\bar{\alpha}_v(\xi_k, \eta_r)$ can be calculated. Most of the integral should be performed by means of a numerical method and a part can be integrated analytically. In the intervals of $\xi_k-\delta \leq \xi_0 \leq \xi_k+\delta$ and $\eta_r - \varepsilon \leq \eta_r + \varepsilon$, we can put

$$e^{-k(R-X)} = 1 - k(R-X) \tag{53}$$

Therefore, the kernel of Eq.(48) becomes

$$Kv_{12}(X,Y) = -X(1+kX)\frac{1}{R^3} - k^2X^2\frac{1}{R^2} + k(1+2kX)\frac{1}{R} - k^2 \tag{54}$$

and we can put

$$\bar{\lambda}_{i,j_1}(\eta_0) = \bar{\gamma}_p(\xi_i, \eta_{j_1}) \tag{55}$$

Consequently, Eq.(52) in the above intervals results in

$$\frac{cs}{2}\sum_{i=1}^{n}\left(\int_0^{\xi_k-\delta}+\int_{\xi_k-\delta}^{\xi_k+\delta}\right)\left[h_i^{(n)}(\xi_0)\sum_{j_1=0}^{m}\int_{\eta_r-\varepsilon}^{\eta_r+\varepsilon}\bar{\lambda}_{i,j_1}(\eta_0)\,Kv_{12}(X,Y)\,d\eta_0\right]d\xi_0$$

$$-\frac{cs}{2}\sum_{i=1}^{n}\left(\int_0^{\xi_k-\delta}+\int_{\xi_k-\delta}^{\xi_k+\delta}\right)\left[h_i^{(n)}(\xi_0)\sum_{j_1=0}^{m}\bar{\gamma}_p(\xi_i,\eta_{j_1})\int_{\eta_r-\varepsilon}^{\eta_r+\varepsilon}\left\{-X(1+kX)\frac{1}{R^3}\right.\right.$$

$$\left.\left.-k^2X^2\frac{1}{R^2}+k(1+2kX)\frac{1}{R}-k^2\right\}d\eta_0\right]d\xi_0$$

$$=c\sum_{i=1}^{n}\left(\int_0^{\xi_k-\delta}+\int_{\xi_k-\delta}^{\xi_k+\delta}\right)\left[h_i^{(n)}(\xi_0)\sum_{j_1=0}^{m}\bar{\gamma}_p(\xi_i,\eta_{j_1})\left\{(1+kX)\frac{s\varepsilon}{X\sqrt{X^2+(\frac{s\varepsilon}{2})^2}}\right.\right.$$

$$\left.\left.+2k^2X\tan^{-1}\frac{s\varepsilon}{2X}-2k(1+2kX)\,\mathrm{cosech}^{-1}\frac{s\varepsilon}{2X}+k^2s\varepsilon\right\}\right]d\xi_0 \tag{56}$$

Especially, in the intervals of $\xi_k-\delta \leq \xi_0 \leq \xi_k+\delta$ and $\eta_r-\varepsilon \leq \eta_r+\varepsilon$ of the above integral, we can put $h_i^{(n)}(\xi_0) = h_i^{(n)}(\xi_k)$, and then evaluating the logarithmic singularities, we obtain

$$\frac{cs}{2}\sum_{i=1}^{n}\int_{\xi_k-\delta}^{\xi_k+\delta}\left[h_i^{(n)}(\xi_0)\sum_{j_1=0}^{m}\int_{\eta_r-\varepsilon}^{\eta_r+\varepsilon}\bar{\lambda}_{i,j_1}(\eta_0)\,Kv_{12}(X,Y)\,d\eta_0\right]d\xi_0$$

$$=4kc\delta\left[\log\frac{2c\delta}{s\varepsilon}-\log\left\{1+\sqrt{1+\left(\frac{2c\delta}{s\varepsilon}\right)^2}\right\}\right]\sum_{i=1}^{n}h_i^{(n)}(\xi_k)\sum_{j_1=0}^{m}\bar{\gamma}_p(\xi_i,\eta_{j_1}) \tag{57}$$

2.3 Lift Coefficient

As mentioned in section 2.2, the resultant loading distribution γ_p calculated in the third step (iii) may be said to be the very loading distribution γ including the effects of viscosity. As a result, the lift coefficient including the effects of viscosity is expressed as follows:

$$C_L = \frac{L}{\frac{1}{2}\rho\bar{U}wcs} = \frac{4\pi}{\alpha}\int_{-1}^{+1}\int_0^{+1}\bar{\gamma}(\xi_0,\eta_0)\,d\xi_0\,d\eta_0 \tag{58}$$

Substitution of Eq.(26) into the above equation yields

$$C_L = \frac{4\pi}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{m}\bar{\gamma}(\xi_i,\eta_j)\,H_i^{(n)}G_j^{(m)} \tag{59}$$

where

$$H_i^{(n)} = \int_0^1 h_i^{(n)}(\xi_0)\,d\xi_0 = \frac{\pi}{2n+1}\sin\theta_i \tag{60}$$

$$G_j^{(m)} = \int_{-1}^{+1} g_j^{(m)}(\eta_0)\,d\eta_0 = \frac{\pi}{m+1}\sin\phi_j \tag{61}$$

3. Results and Discussions

The effects of viscosity and aspect ratio on the lifting force applied on the finite span rectangular flat plate are represented in Fig.2, where the calculations were carried out for the cases of the chord length $c=1$, span $s=2$, aspect ratios $A=s/c$ =2, 4, 8, 16, Reynolds numbers $R_e = kc = Uc/(2\nu)$=10, 100, 1000, ∞ (invicid). The computational procedures were as follows:

The numbers of the loading points were (n,m) = (2,4) when A=2, (2,8) when A=4, (2,20) when A=8, (2,40) when A=16. $\bar{\alpha}(\xi_k, \eta_r)=1$ were put for all the downwash points. The simultaneous linear equations (29) including nm unknowns of $\bar{\gamma}_p(\xi_i, \eta_j)$ were solved by means of the sweep-out method. The numerical integrations by the Simpson method were applied to Eqs.(35), (52) and (56), and the errors for the integrals were kept down in less than 0.1%. In Eq.(52), there may be a value of h that $K_{v12}(X,Y)$ approaches to nearly zero for the range of $\frac{s}{2}|\eta_r - \eta_0| > h$. Such values of h were numerically investigated and as the results showed that h=0.5, 0.3, 0.1 against R_e =10, 100, 1000, respectively. Now that the values of h are not so great and that K_{v12} approaches rapidly to zero in the vicinity of $Y=h$, actually, the satisfactory results were obtained by calculation only in case of $\eta_j = \eta_r$, and saving of time for computating was attained. δ and ε contained in Eqs.(52), (56), (57) were taken as $c\delta = s\varepsilon =$ 0.0001, 0.00001, 0.000003 for R_e =10, 100, 1000, respectively, after discussions about their values.

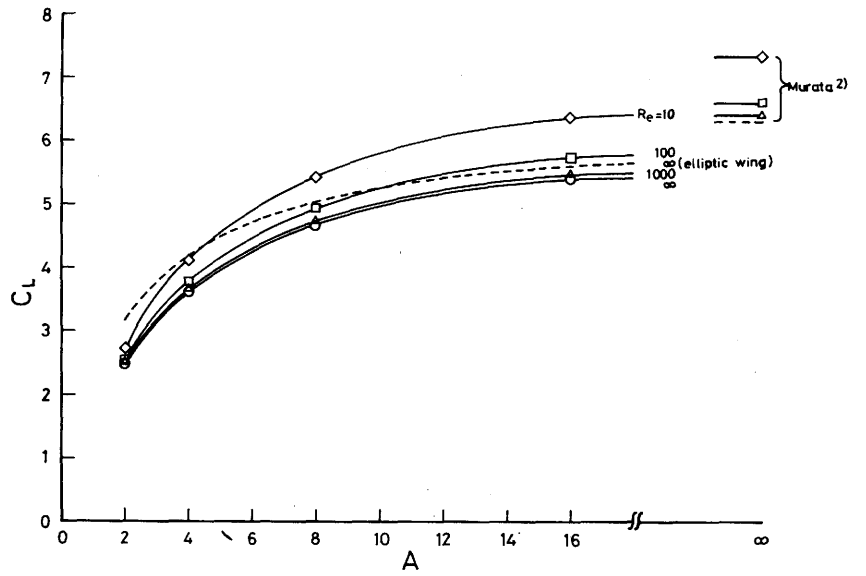

Fig.2 Effects of viscosity and aspect ratio on lifting force

The computational results for the case of two-dimensional flat plate by S.Murata et al.[2] are also shown in Fig.2 for a reference. As everybody knows, the relationship between the aspect ratio A and lift coefficient C_L' of the elliptic wing in the invicid flow is of the form

$$C_L' = \frac{2\pi}{1+\frac{2}{A}} \tag{62}$$

and the computational results are also shown in Fig.2.

The discussions about Fig.2 lead to the conclusions as follows:

(1) The lift coefficient of the rectangular flat plate increases with higher viscosity (or with lower value of Re).

(2) At the same value of Re, the value of C_L increases with the aspect ratio A and approaches gradually to a constant.

(3) In case of invicid flow, the lift coefficient C_L' of the elliptic wing approaches to C_L of the rectangular flat plate with increase of A.

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